Theory –

The starting equation for the Weighted Least Squares (WLS) state estimation algorithm is:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} = \begin{bmatrix} h_1(x_1, x_2, \dots x_n) \\ h_2(x_1, x_2, \dots x_n) \\ \vdots \\ h_m(x_1, x_2, \dots x_n) \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} = h(x) + e \qquad (1)$$

The vector z of m measured values is:

$$z^T = [z_1, z_2, \dots z_m]$$

The vector h:

$$h^T = [h_1(x), h_2(x), \dots h_m(x)]$$

containing the non-linear functions $h_i(x)$ relates the predicted value of measurement i to the state vector x containing n variables. The state vector x is defined as:

$$x^T = [x_1, x_2, \dots x_n]$$

And e is the vector of measurement errors:

$$e^T = [e_1, e_2, \dots e_m]$$

The measurement errors e_i are assumed to satisfy the following statistical properties:

1. Zero Mean:

$$E(e_i) = 0, \ for\ i = 1, \dots, m \qquad (2)$$

2. Independence:

$$E(e_i e_j) = 0\ for\ i \neq j \qquad (3)$$

And the covariance matrix is diagonal:

$$Cov(e) = E(e.e^T) = R = diag\{\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2\} \qquad (4)$$

Objective Function

The objective function is given by the following relation:

$$J(x) = \sum_{i=1}^{m} \frac{(z_i - h_i(x))^2}{R_{ii}} = [z - h(x)]^T R^{-1} [z - h(x)] \qquad (5)$$

Minimization Condition

The minimization condition is:

$$g(x) = \frac{\partial J(x)}{\partial x} = -H^T(x)R^{-1}[z - h(x)] = 0$$

where, $H(x) = \frac{\partial h(x)}{\partial x}$

Expanding $g(x)$ into its Taylor series gives:

$$g(x) = g(x^k) + G(x^k)(x - x^k) + \ldots = 0 \qquad (6)$$

The $(k + 1)$ iterate is related to the k-th iterate via:

$$x^{k+1} = x^k - G(x^k)^{-1}g(x^k)$$

where the gain matrix is:

$$G(x^k) = \frac{\partial g(x^k)}{\partial x} = H^T(x^k)R^{-1}H(x^k)$$

And

$$g(x^k) = [H]^T(x^k)R^{-1}[z - h(x^k)]$$

Normal Equation and Solution Procedure

The normal equation for the state estimation calculation follows from the minimization condition and is given by:

$$G(x^k)\Delta x^{k+1} = H^T(x^k)R^{-1}\left(z - h(x^k)\right) \qquad (7)$$

where $\Delta x^{k+1} = x^{k+1} - x^k$

The WLS SE algorithm steps are as follows:

1. Set $k = 0$
2. Initialize the state vector x^k, typically a flat start
3. Calculate the measurement function $h(x^k)$
4. Build the measurement Jacobian $H(x^k)$
5. Calculate the gain matrix $G(x^k) = H^T(x^k)R^{-1}H(x^k)$
6. Calculate the right-hand side (RHS) of the normal equation $H^T(x^k)R^{-1}(z - h(x^k))$
7. Solve the normal equation (7) for Δx^k
8. Check for convergence using $max|\Delta x^k| \leq \in$
9. If not converged, update $\Delta x^{k+1} = x^k + \Delta x^k$ and go to step 3. Otherwise, stop.